



# Syllabus

## Cambridge International AS & A Level Further Mathematics 9231

For examination in June and November 2023, 2024 and 2025.



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## Why choose Cambridge?

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Cambridge International prepares school students for life, helping them develop an informed curiosity and a lasting passion for learning. We are part of the University of Cambridge.

Our Cambridge Pathway gives students a clear path for educational success from age 5 to 19. Schools can shape the curriculum around how they want students to learn – with a wide range of subjects and flexible ways to offer them. It helps students discover new abilities and a wider world, and gives them the skills they need for life, so they can achieve at school, university and work.

Our programmes and qualifications set the global standard for international education. They are created by subject experts, rooted in academic rigour and reflect the latest educational research. They provide a strong platform for students to progress from one stage to the next, and are well supported by teaching and learning resources.

We review all our syllabuses regularly, so they reflect the latest research evidence and professional teaching practice – and take account of the different national contexts in which they are taught.

We consult with teachers to help us design each syllabus around the needs of their learners. Consulting with leading universities has helped us make sure our syllabuses encourage students to master the key concepts in the subject and develop the skills necessary for success in higher education.

Our mission is to provide educational benefit through provision of international programmes and qualifications for school education and to be the world leader in this field. Together with schools, we develop Cambridge learners who are confident, responsible, reflective, innovative and engaged – equipped for success in the modern world.

Every year, nearly a million Cambridge students from 10 000 schools in 160 countries prepare for their future with the Cambridge Pathway.

**'We think the Cambridge curriculum is superb preparation for university.'**

**Christoph Guttentag**, Dean of Undergraduate Admissions, Duke University, USA



### Quality management

Cambridge International is committed to providing exceptional quality. In line with this commitment, our quality management system for the provision of international qualifications and education programmes for students aged 5 to 19 is independently certified as meeting the internationally recognised standard, ISO 9001:2015. Learn more at [www.cambridgeinternational.org/ISO9001](http://www.cambridgeinternational.org/ISO9001)

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## **Important: Changes to this syllabus**

**For information about changes to this syllabus for 2023, 2024 and 2025, go to page 47.**

The latest syllabus is version 1, published September 2020. There are no significant changes which affect teaching.

Any textbooks endorsed to support the syllabus for examination from 2020 are still suitable for use with this syllabus.

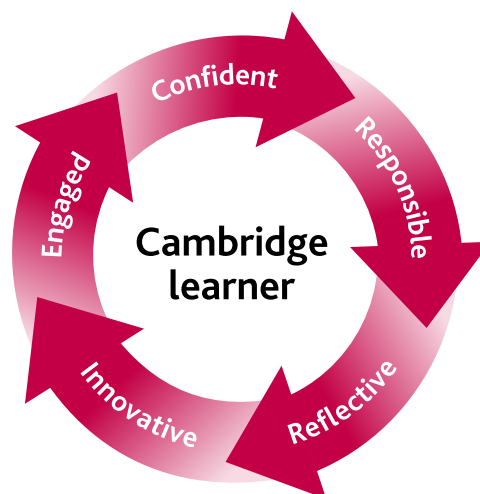
# 1 Why choose this syllabus?

## Key benefits

The best motivation for a student is a real passion for the subject they're learning. By offering students a variety of Cambridge International AS & A Levels, you can give them the greatest chance of finding the path of education they most want to follow. With over 50 subjects to choose from, students can select the ones they love and that they're best at, which helps motivate them throughout their studies.

Following a Cambridge International AS & A Level programme helps students develop abilities which universities value highly, including:

- a deep understanding of their subjects
- higher order thinking skills – analysis, critical thinking, problem solving
- presenting ordered and coherent arguments
- independent learning and research.



**Cambridge International AS & A Level Further Mathematics** develops a set of transferable skills. These include the skill of working with mathematical information, as well as the ability to think logically and independently, consider accuracy, model situations mathematically, analyse results and reflect on findings. Learners can apply these skills across a wide range of subjects and the skills equip them well for progression to higher education or directly into employment. Learners will find that the additional time spent studying this subject will support their understanding of A Level Mathematics.

Our approach in Cambridge International AS & A Level Further Mathematics encourages learners to be:

**confident**, using and sharing information and ideas, and using mathematical techniques to solve problems. These skills build confidence and support work in other subject areas as well as in mathematics.

**responsible**, through learning and applying skills which prepare them for future academic studies, helping them to become numerate members of society.

**reflective**, through making connections between different branches of mathematics and considering the outcomes of mathematical problems and modelling.

**innovative**, through solving both familiar and unfamiliar problems in different ways, selecting from a range of mathematical and problem-solving techniques.

**engaged**, by the beauty and structure of mathematics, its patterns and its many applications to real life situations.

**'Cambridge students develop a deep understanding of subjects and independent thinking skills.'**

Tony Hines, Principal, Rockledge High School, USA

## Key concepts

Key concepts are essential ideas that help students develop a deep understanding of their subject and make links between different aspects. Key concepts may open up new ways of thinking about, understanding or interpreting the important things to be learned.

Good teaching and learning will incorporate and reinforce a subject's key concepts to help students gain:

- a greater depth as well as breadth of subject knowledge
- confidence, especially in applying knowledge and skills in new situations
- the vocabulary to discuss their subject conceptually and show how different aspects link together
- a level of mastery of their subject to help them enter higher education.

The key concepts identified below, carefully introduced and developed, will help to underpin the course you will teach. You may identify additional key concepts which will also enrich teaching and learning.

The key concepts for Cambridge International AS & A Level Further Mathematics are:

- **Problem solving**

Mathematics is fundamentally problem solving and representing systems and models in different ways. These include:

- Algebra: this is an essential tool which supports and expresses mathematical reasoning and provides a means to generalise across a number of contexts.
- Geometrical techniques: algebraic representations also describe a spatial relationship, which gives us a new way to understand a situation.
- Calculus: this is a fundamental element which describes change in dynamic situations and underlines the links between functions and graphs.
- Mechanical models: these explain and predict how particles and objects move or remain stable under the influence of forces.
- Statistical methods: these are used to quantify and model aspects of the world around us. Probability theory predicts how chance events might proceed, and whether assumptions about chance are justified by evidence.

- **Communication**

Mathematical proof and reasoning is expressed using algebra and notation so that others can follow each line of reasoning and confirm its completeness and accuracy. Mathematical notation is universal. Each solution is structured, but proof and problem solving also invite creative and original thinking.

- **Mathematical modelling**

Mathematical modelling can be applied to many different situations and problems, leading to predictions and solutions. A variety of mathematical content areas and techniques may be required to create the model. Once the model has been created and applied, the results can be interpreted to give predictions and information about the real world.

## Recognition and progression

Our expertise in curriculum, teaching and learning, and assessment is the basis for the recognition of our programmes and qualifications around the world. Every year thousands of students with Cambridge International AS & A Levels gain places at leading universities worldwide. Our programmes and qualifications are valued by top universities around the world including those in the UK, US (including Ivy League universities), Europe, Australia, Canada and New Zealand.

UK NARIC, the national agency in the UK for the recognition and comparison of international qualifications and skills, has carried out an independent benchmarking study of Cambridge International AS & A Level and found it to be comparable to the standard of AS & A Level in the UK. This means students can be confident that their Cambridge International AS & A Level qualifications are accepted as equivalent, grade for grade, to UK AS & A Levels by leading universities worldwide.

Cambridge International AS Level Further Mathematics makes up the first half of the Cambridge International A Level course in further mathematics and provides a foundation for the study of further mathematics at Cambridge International A Level. Depending on local university entrance requirements, students may be able to use it to progress directly to university courses in mathematics or some other subjects. It is also suitable as part of a course of general education.

Cambridge International A Level Further Mathematics provides an excellent foundation for the study of mathematics or related courses in higher education. Equally it is suitable as part of a course of general education.

For more information about the relationship between the Cambridge International AS Level and Cambridge International A Level see the 'Assessment overview' section of the Syllabus overview.

We recommend learners check the Cambridge recognitions database and the university websites to find the most up-to-date entry requirements for courses they wish to study.

Learn more at [www.cambridgeinternational.org/recognition](http://www.cambridgeinternational.org/recognition)

**'The depth of knowledge displayed by the best A Level students makes them prime targets for America's Ivy League universities'**

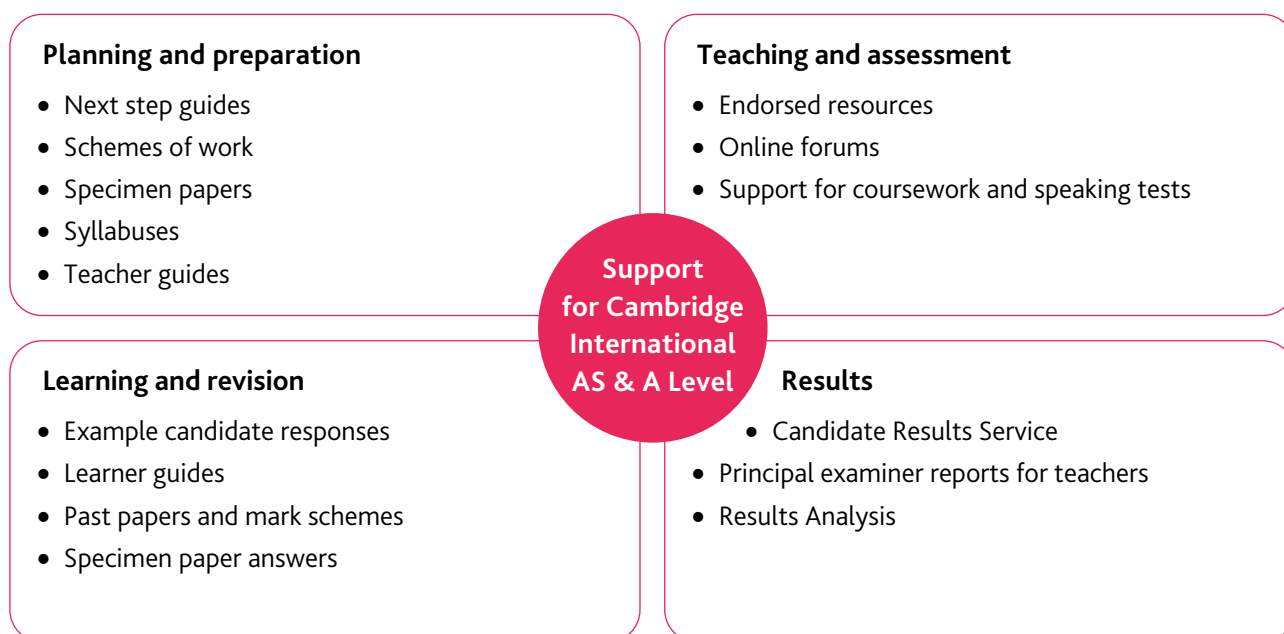
Yale University, USA

## Supporting teachers

We provide a wide range of resources, detailed guidance and innovative training and professional development so that you can give your students the best possible preparation for Cambridge International AS & A Level. To find out which resources are available for each syllabus go to our School Support Hub.

The School Support Hub is our secure online site for Cambridge teachers where you can find the resources you need to deliver our programmes. You can also keep up to date with your subject and the global Cambridge community through our online discussion forums.

Find out more at [www.cambridgeinternational.org/support](http://www.cambridgeinternational.org/support)



Sign up for email notifications about changes to syllabuses, including new and revised products and services at [www.cambridgeinternational.org/syllabusupdates](http://www.cambridgeinternational.org/syllabusupdates)

## Professional development

We support teachers through:

- Introductory Training – face-to-face or online
- Extension Training – face-to-face or online
- Enrichment Professional Development – face-to-face or online

Find out more at [www.cambridgeinternational.org/events](http://www.cambridgeinternational.org/events)

- Cambridge Professional Development Qualifications

Find out more at [www.cambridgeinternational.org/profdev](http://www.cambridgeinternational.org/profdev)



### Supporting exams officers

We provide comprehensive support and guidance for all Cambridge exams officers. Find out more at: [www.cambridgeinternational.org/eoguide](http://www.cambridgeinternational.org/eoguide)

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## 2 Syllabus overview

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### Aims

The aims describe the purposes of a course based on this syllabus.

The aims are to enable students to:

- further develop their mathematical knowledge and skills in a way which encourages confidence and provides satisfaction and enjoyment
- develop a greater understanding of mathematical principles and a further appreciation of mathematics as a logical and coherent subject
- acquire a greater range of mathematical skills, particularly those which will enable them to use applications of mathematics in the context of everyday situations and of other subjects they may be studying
- further develop the ability to analyse problems logically
- recognise when and how a situation may be represented mathematically, identify and interpret relevant factors and select an appropriate mathematical method to solve the problem
- use mathematics fluently as a means of communication with emphasis on the use of clear expression
- acquire the mathematical background necessary for further study in mathematics or related subjects.

Cambridge Assessment International Education is an education organisation and politically neutral. The contents of this syllabus, examination papers and associated materials do not endorse any political view. We endeavour to treat all aspects of the exam process neutrally.





## Content overview

| Content section                    | Assessment component | Topics included   |
|------------------------------------|----------------------|---|
| 1 Further Pure Mathematics 1       | Paper 1              | 1.1 Roots of polynomial equations<br>1.2 Rational functions and graphs<br>1.3 Summation of series<br>1.4 Matrices<br>1.5 Polar coordinates<br>1.6 Vectors<br>1.7 Proof by induction |
| 2 Further Pure Mathematics 2       | Paper 2              | 2.1 Hyperbolic functions<br>2.2 Matrices<br>2.3 Differentiation<br>2.4 Integration<br>2.5 Complex numbers<br>2.6 Differential equations   |
| 3 Further Mechanics                | Paper 3              | 3.1 Motion of a projectile<br>3.2 Equilibrium of a rigid body<br>3.3 Circular motion<br>3.4 Hooke's law<br>3.5 Linear motion under a variable force<br>3.6 Momentum                 |
| 4 Further Probability & Statistics | Paper 4              | 4.1 Continuous random variables<br>4.2 Inference using normal and $t$ -distributions<br>4.3 $\chi^2$ -tests<br>4.4 Non-parametric tests<br>4.5 Probability generating functions     |

'Cambridge International AS & A Levels prepare students well for university because they've learnt to go into a subject in considerable depth. There's that ability to really understand the depth and richness and the detail of a subject. It's a wonderful preparation for what they are going to face at university.'

US Higher Education Advisory Council

## Structure

There are four components that can be combined in specific ways (please see below):

Paper 1: Further Pure Mathematics 1

Paper 2: Further Pure Mathematics 2

Paper 3: Further Mechanics

Paper 4: Further Probability & Statistics

All AS Level candidates take two written papers.

All A Level candidates take four written papers.

### AS Level Further Mathematics

The Cambridge International AS Level Further Mathematics qualification offers two different options:

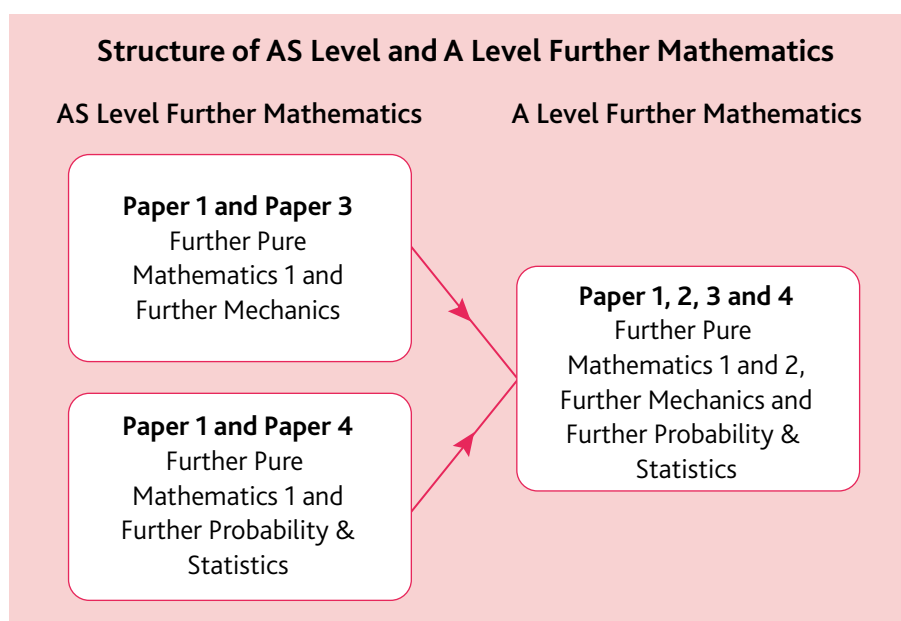
- Further Pure Mathematics 1 and Further Mechanics (Paper 1 and Paper 3)
- or
- Further Pure Mathematics 1 and Further Probability & Statistics (Paper 1 and Paper 4).

### A Level Further Mathematics

Cambridge International A Level Further Mathematics includes all four components:

- Paper 1: Further Pure Mathematics 1
- Paper 2: Further Pure Mathematics 2
- Paper 3: Further Mechanics
- Paper 4: Further Probability & Statistics.

See page 10 for a table showing all possible assessment routes.



Please see the School Support Hub at [www.cambridgeinternational.org/support](http://www.cambridgeinternational.org/support) for more information about parallel teaching of this syllabus alongside Cambridge International AS & A Level Mathematics and recommended routes through the Cambridge International AS & A Level Mathematics qualifications 9709 and 9231.

## Assessment overview

### Paper 1

Further Pure Mathematics 1                      2 hours  
 75 marks  
 6 to 8 structured questions based on the  
 Further Pure Mathematics 1 subject content  
 Answer all questions  
 Written examination  
 Externally assessed  
 60% of the AS Level  
 30% of the A Level  
 Compulsory for AS Level and A Level

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### Paper 2

Further Pure Mathematics 2                      2 hours  
 75 marks  
 7 to 9 structured questions based on the  
 Further Pure Mathematics 2 subject content  
 Answer all questions  
 Written examination  
 Externally assessed  
 30% of the A Level only  
 Compulsory for A Level

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### Paper 3

Further Mechanics                                  1 hour 30 minutes  
 50 marks  
 5 to 7 structured questions based on the  
 Further Mechanics subject content  
 Answer all questions  
 Written examination  
 Externally assessed  
 40% of the AS Level  
 20% of the A Level  
 Offered as part of AS Level or A Level

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### Paper 4

Further Probability &                              1 hour 30 minutes  
 Statistics  
 50 marks  
 5 to 7 structured questions based on the  
 Further Probability & Statistics subject content  
 Answer all questions  
 Written examination  
 Externally assessed  
 40% of the AS Level  
 20% of the A Level  
 Offered as part of AS Level or A Level

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### Three routes for Cambridge International AS & A Level Further Mathematics

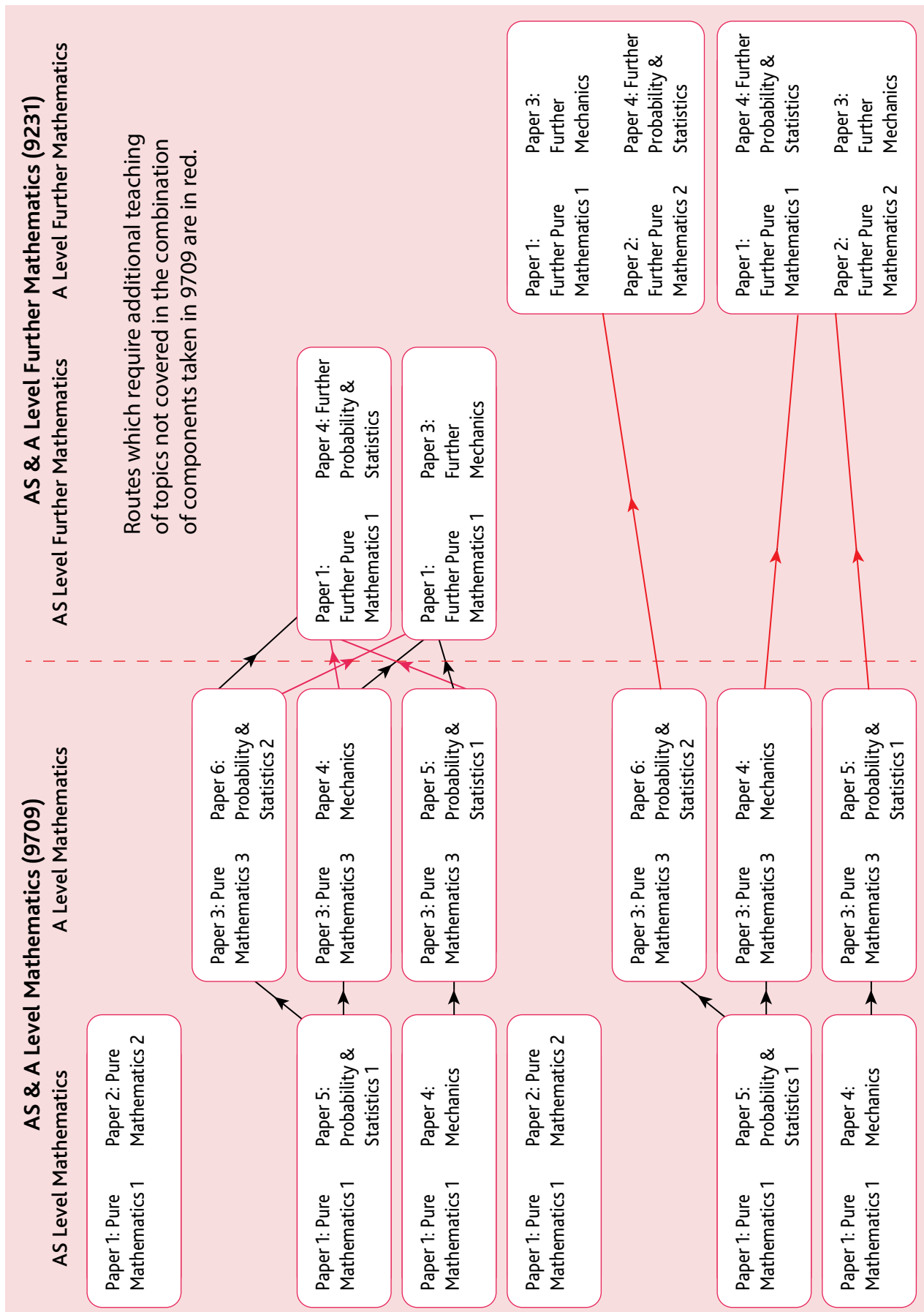
Candidates may combine components as shown below.

| <b>Route 1</b><br><b>AS Level only</b><br>(Candidates take the AS components in the same series) | <b>Paper 1</b><br><b>Further Pure Mathematics 1</b> | <b>Paper 2</b><br><b>Further Pure Mathematics 2</b> | <b>Paper 3</b><br><b>Further Mechanics</b> | <b>Paper 4</b><br><b>Further Probability &amp; Statistics</b> |
|--|---|---|--|---|
| <b>Either</b>  | ✓   | Not available for AS Level                          | ✓  |   |
| <b>Or</b>  | ✓   |   |  | ✓   |

| <b>Route 2</b><br><b>A Level (staged over two years)</b> | <b>Paper 1</b><br><b>Further Pure Mathematics 1</b> | <b>Paper 2</b><br><b>Further Pure Mathematics 2</b> | <b>Paper 3</b><br><b>Further Mechanics</b> | <b>Paper 4</b><br><b>Further Probability &amp; Statistics</b> |
|--|---|---|--|---|
| <b>Either</b><br>Year 1 AS Level                         | ✓   |   | ✓  |   |
| Year 2 Complete the A Level                              |   | ✓   |  | ✓   |
| <b>Or</b><br>Year 1 AS Level                             | ✓   |   |  | ✓   |
| Year 2 Complete the A Level                              |   | ✓   | ✓  |   |

| <b>Route 3</b><br><b>A Level</b><br>(Candidates take the A Level components in the same series) | <b>Paper 1</b><br><b>Further Pure Mathematics 1</b> | <b>Paper 2</b><br><b>Further Pure Mathematics 2</b> | <b>Paper 3</b><br><b>Further Mechanics</b> | <b>Paper 4</b><br><b>Further Probability &amp; Statistics</b> |
|---|---|---|--|---|
| Year 2 full A Level   | ✓   | ✓   | ✓  | ✓   |

# Routes through AS & A Level Mathematics and AS & A Level Further Mathematics



## Assessment objectives

The assessment objectives (AOs) are:

### AO1 Knowledge and understanding

- Show understanding of relevant mathematical concepts, terminology and notation
- Recall accurately and use appropriate mathematical manipulative techniques

### AO2 Application and communication

- Recognise the appropriate mathematical procedure for a given situation
- Apply appropriate combinations of mathematical skills and techniques in solving problems
- Present relevant mathematical work, and communicate corresponding conclusions, in a clear and logical way

### Weighting for assessment objectives

The approximate weightings ( $\pm 5\%$ ) allocated to each of the assessment objectives (AOs) are summarised below.

Assessment objectives as an approximate percentage of each component

| Assessment objective              | Weighting in components % |         |         |         |
|-----------------------------------|---------------------------|---------|---------|---------|
|                                   | Paper 1                   | Paper 2 | Paper 3 | Paper 4 |
| AO1 Knowledge and understanding   | 45                        | 45      | 45      | 45      |
| AO2 Application and communication | 55                        | 55      | 55      | 55      |

Assessment objectives as an approximate percentage of each qualification

| Assessment objective              | Weighting in AS Level % | Weighting in A Level % |
|-----------------------------------|-------------------------|------------------------|
| AO1 Knowledge and understanding   | 45                      | 45                     |
| AO2 Application and communication | 55                      | 55                     |

## 3 Subject content

This syllabus gives you the flexibility to design a course that will interest, challenge and engage your learners. Where appropriate you are responsible for selecting resources and examples to support your learners' study. These should be appropriate for the learners' age, cultural background and learning context as well as complying with your school policies and local legal requirements.

The mathematical content for each component is detailed below. You can teach the topics in any order you find appropriate. However, please note the prior knowledge requirements below, and the information about calculator use found in 4 Details of the assessment.

Notes and examples are included to clarify the syllabus content. Please note that these are examples only and examination questions may differ from the examples given.

### Prior knowledge

It is expected that learners will have studied the majority of the Cambridge International AS & A Level Mathematics (9709) subject content before studying Cambridge International AS & A Level Further Mathematics (9231).

The prior knowledge required for each Further Mathematics component is shown in the following table.

| Component in AS & A Level Further Mathematics (9231) | Prior knowledge required from AS & A Level Mathematics (9709)               |
|--|---|
| 9231 Paper 1:<br>Further Pure Mathematics 1          | <ul style="list-style-type: none"> <li>9709 Papers 1 and 3</li> </ul>       |
| 9231 Paper 2:<br>Further Pure Mathematics 2          | <ul style="list-style-type: none"> <li>9709 Papers 1 and 3</li> </ul>       |
| 9231 Paper 3:<br>Further Mechanics                   | <ul style="list-style-type: none"> <li>9709 Papers 1, 3 and 4</li> </ul>    |
| 9231 Paper 4:<br>Further Probability & Statistics    | <ul style="list-style-type: none"> <li>9709 Papers 1, 3, 5 and 6</li> </ul> |

Please see the School Support Hub at [www.cambridgeinternational.org/support](http://www.cambridgeinternational.org/support) for more information about parallel teaching of Cambridge International AS & A Level Mathematics (9709) and AS & A Level Further Mathematics (9231).

The support document *Guide to prior learning for Paper 4 Further Probability & Statistics* can be found on the Cambridge International website.

# 1 Further Pure Mathematics 1 (for Paper 1)

## 1.1 Roots of polynomial equations

Candidates should be able to:

- recall and use the relations between the roots and coefficients of polynomial equations
- use a substitution to obtain an equation whose roots are related in a simple way to those of the original equation.

Notes and examples

e.g. to evaluate symmetric functions of the roots or to solve problems involving unknown coefficients in equations; restricted to equations of degree 2, 3 or 4 only.

Substitutions will not be given for the easiest cases, e.g. where the new roots are reciprocals or squares or a simple linear function of the old roots.

## 1.2 Rational functions and graphs

Candidates should be able to:

- sketch graphs of simple rational functions, including the determination of oblique asymptotes, in cases where the degree of the numerator and the denominator are at most 2
- understand and use relationships between the graphs of  $y = f(x)$ ,  $y^2 = f(x)$ ,  $y = \frac{1}{f(x)}$ ,  $y = |f(x)|$  and  $y = f(|x|)$ .

Notes and examples

Including determination of the set of values taken by the function, e.g. by the use of a discriminant.

Detailed plotting of curves will not be required, but sketches will generally be expected to show significant features, such as turning points, asymptotes and intersections with the axes.

Including use of such sketch graphs in the course of solving equations or inequalities.

## 1.3 Summation of series

Candidates should be able to:

- use the standard results for  $\sum r$ ,  $\sum r^2$ ,  $\sum r^3$  to find related sums
- use the method of differences to obtain the sum of a finite series
- recognise, by direct consideration of a sum to  $n$  terms, when a series is convergent, and find the sum to infinity in such cases.

Notes and examples

Use of partial fractions to express a general term in a suitable form may be required.



## 1 Further Pure Mathematics 1

### 1.4 Matrices

#### Candidates should be able to:

- carry out operations of matrix addition, subtraction and multiplication, and recognise the terms zero matrix and identity (or unit) matrix
- recall the meaning of the terms 'singular' and 'non-singular' as applied to square matrices and, for  $2 \times 2$  and  $3 \times 3$  matrices, evaluate determinants and find inverses of non-singular matrices
- understand and use the result, for non-singular matrices,  $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- understand the use of  $2 \times 2$  matrices to represent certain geometric transformations in the  $x$ - $y$  plane, in particular
  - understand the relationship between the transformations represented by  $\mathbf{A}$  and  $\mathbf{A}^{-1}$
  - recognise that the matrix product  $\mathbf{AB}$  represents the transformation that results from the transformation represented by  $\mathbf{B}$  followed by the transformation represented by  $\mathbf{A}$
  - recall how the area scale factor of a transformation is related to the determinant of the corresponding matrix
  - find the matrix that represents a given transformation or sequence of transformations
- understand the meaning of 'invariant' as applied to points and lines in the context of transformations represented by matrices, and solve simple problems involving invariant points and invariant lines.

#### Notes and examples

Including non-square matrices. Matrices will have at most 3 rows and columns.

The notations  $\det \mathbf{M}$  for the determinant of a matrix  $\mathbf{M}$ , and  $\mathbf{I}$  for the identity matrix, will be used.

Extension to the product of more than two matrices may be required.

Understanding of the terms 'rotation', 'reflection', 'enlargement', 'stretch' and 'shear' for 2D transformations will be required.

Other 2D transformations may be included, but no particular knowledge of them is expected.

e.g. to locate the invariant points of the transformation represented by  $\begin{pmatrix} 6 & 5 \\ 2 & 3 \end{pmatrix}$ , or to find the invariant lines through the origin for  $\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$ , or to show that any line with gradient 1 is invariant for  $\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ .

## 1 Further Pure Mathematics 1

### 1.5 Polar coordinates

#### Candidates should be able to:

- understand the relations between Cartesian and polar coordinates, and convert equations of curves from Cartesian to polar form and vice versa
- sketch simple polar curves, for  $0 \leq \theta < 2\pi$  or  $-\pi < \theta \leq \pi$  or a subset of either of these intervals
- recall the formula  $\frac{1}{2} \int r^2 d\theta$  for the area of a sector, and use this formula in simple cases.

#### Notes and examples

The convention  $r \geq 0$  will be used.

Detailed plotting of curves will not be required, but sketches will generally be expected to show significant features, such as symmetry, coordinates of intersections with the initial line, the form of the curve at the pole and least/greatest values of  $r$ .

### 1.6 Vectors

#### Candidates should be able to:

- use the equation of a plane in any of the forms  $ax + by + cz = d$  or  $\mathbf{r} \cdot \mathbf{n} = p$  or  $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$  and convert equations of planes from one form to another as necessary in solving problems
- recall that the vector product  $\mathbf{a} \times \mathbf{b}$  of two vectors can be expressed either as  $|\mathbf{a}||\mathbf{b}|\sin \theta \hat{\mathbf{n}}$ , where  $\hat{\mathbf{n}}$  is a unit vector, or in component form as  $(a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}$
- use equations of lines and planes, together with scalar and vector products where appropriate, to solve problems concerning distances, angles and intersections, including
  - determining whether a line lies in a plane, is parallel to a plane or intersects a plane, and finding the point of intersection of a line and a plane when it exists
  - finding the foot of the perpendicular from a point to a plane
  - finding the angle between a line and a plane, and the angle between two planes
  - finding an equation for the line of intersection of two planes
  - calculating the shortest distance between two skew lines
  - finding an equation for the common perpendicular to two skew lines.

#### Notes and examples

## 1 Further Pure Mathematics 1

### 1.7 Proof by induction

Candidates should be able to:

- use the method of mathematical induction to establish a given result
- recognise situations where conjecture based on a limited trial followed by inductive proof is a useful strategy, and carry this out in simple cases.

Notes and examples

e.g.  $\sum_{r=1}^n r^4 = \frac{1}{4}n^2(n+1)^2,$

$u_n = \frac{1}{2}(1 + 3^{n-1})$  for the sequence given by

$u_{n+1} = 3u_n - 1$  and  $u_1 = 1,$

$$\begin{pmatrix} 4 & -1 \\ 6 & -1 \end{pmatrix}^n = \begin{pmatrix} 3 \times 2^n - 2 & 1 - 2^n \\ 3 \times 2^{n+1} - 6 & 3 - 2^{n+1} \end{pmatrix},$$

$3^{2n} + 2 \times 5^n - 3$  is divisible by 8.

e.g. find the  $n$ th derivative of  $x e^x,$

find  $\sum_{r=1}^n r \times r!.$

## 2 Further Pure Mathematics 2 (for Paper 2)

Knowledge of Paper 1: Further Pure Mathematics 1 subject content from this syllabus is assumed for this component.

### 2.1 Hyperbolic functions

Candidates should be able to:

- understand the definitions of the hyperbolic functions  $\sinh x$ ,  $\cosh x$ ,  $\tanh x$ ,  $\operatorname{sech} x$ ,  $\operatorname{cosech} x$ ,  $\operatorname{coth} x$  in terms of the exponential function
- sketch the graphs of hyperbolic functions
- prove and use identities involving hyperbolic functions
- understand and use the definitions of the inverse hyperbolic functions and derive and use the logarithmic forms.

Notes and examples

e.g.  $\cosh^2 x - \sinh^2 x = 1$ ,  $\sinh 2x = 2 \sinh x \cosh x$ , and similar results corresponding to the standard trigonometric identities.

### 2.2 Matrices

Candidates should be able to:

- formulate a problem involving the solution of 3 linear simultaneous equations in 3 unknowns as a problem involving the solution of a matrix equation, or vice versa
- understand the cases that may arise concerning the consistency or inconsistency of 3 linear simultaneous equations, relate them to the singularity or otherwise of the corresponding matrix, solve consistent systems, and interpret geometrically in terms of lines and planes
- understand the terms 'characteristic equation', 'eigenvalue' and 'eigenvector', as applied to square matrices
- find eigenvalues and eigenvectors of  $2 \times 2$  and  $3 \times 3$  matrices
- express a square matrix in the form  $\mathbf{QDQ}^{-1}$ , where  $\mathbf{D}$  is a diagonal matrix of eigenvalues and  $\mathbf{Q}$  is a matrix whose columns are eigenvectors, and use this expression
- use the fact that a square matrix satisfies its own characteristic equation.

Notes and examples

e.g. three planes meeting in a common point, or in a common line, or having no common points.

Including use of the definition  $\mathbf{Ae} = \lambda\mathbf{e}$  to prove simple properties, e.g. that  $\lambda^n$  is an eigenvalue of  $\mathbf{A}^n$ .

Restricted to cases where the eigenvalues are real and distinct.

e.g. in calculating powers of  $2 \times 2$  or  $3 \times 3$  matrices.

e.g. in finding successive powers of a matrix or finding an inverse matrix; restricted to  $2 \times 2$  or  $3 \times 3$  matrices only.

## 2 Further Pure Mathematics 2

### 2.3 Differentiation

Candidates should be able to:

- differentiate hyperbolic functions and differentiate  $\sin^{-1}x$ ,  $\cos^{-1}x$ ,  $\sinh^{-1}x$ ,  $\cosh^{-1}x$  and  $\tanh^{-1}x$
- obtain an expression for  $\frac{d^2y}{dx^2}$  in cases where the relation between  $x$  and  $y$  is defined implicitly or parametrically
- derive and use the first few terms of a Maclaurin's series for a function.

Notes and examples

Derivation of a general term is not included, but successive 'implicit' differentiation steps may be required, e.g. for  $y = \tan x$  following an initial differentiation rearranged as  $y' = 1 + y^2$ .

### 2.4 Integration

Candidates should be able to:

- integrate hyperbolic functions and recognise integrals of functions of the form  $\frac{1}{\sqrt{a^2 - x^2}}$ ,  $\frac{1}{\sqrt{x^2 + a^2}}$  and  $\frac{1}{\sqrt{x^2 - a^2}}$ , and integrate associated functions using trigonometric or hyperbolic substitutions as appropriate
- derive and use reduction formulae for the evaluation of definite integrals
- understand how the area under a curve may be approximated by areas of rectangles, and use rectangles to estimate or set bounds for the area under a curve or to derive inequalities or limits concerning sums
- use integration to find
  - arc lengths for curves with equations in Cartesian coordinates, including the use of a parameter, or in polar coordinates
  - surface areas of revolution about one of the axes for curves with equations in Cartesian coordinates, including the use of a parameter.

Notes and examples

Including use of completing the square where necessary, e.g. to integrate  $\frac{1}{\sqrt{x^2 + x}}$ .

$$\text{e.g. } \int_0^{\frac{1}{2}\pi} \sin^n x \, dx, \int_0^1 e^{-x}(1-x)^n \, dx.$$

In harder cases hints may be given, e.g.  $\int_0^{\frac{1}{4}\pi} \sec^n x \, dx$

by considering  $\frac{d}{dx}(\tan x \sec^n x)$ .

Questions may involve either rectangles of unit width or rectangles whose width can tend to zero,

$$\text{e.g. } 1 + \ln n > \sum_{r=1}^n \frac{1}{r} > \ln(n+1),$$

$$\sum_{r=1}^n \frac{1}{n} \left(1 + \frac{r}{n}\right)^{-1} \approx \int_0^1 (1+x)^{-1} \, dx.$$

Any questions involving integration may require techniques from Cambridge International A Level Mathematics (9709) applied to more difficult cases, e.g. integration by parts for  $\int e^x \sin x \, dx$ , or use of the substitution  $t = \tan \frac{1}{2}x$ .

Surface areas of revolution for curves with equations in polar coordinates will not be required.

## 2 Further Pure Mathematics 2

### 2.5 Complex numbers

#### Candidates should be able to:

- understand de Moivre's theorem, for a positive or negative integer exponent, in terms of the geometrical effect of multiplication and division of complex numbers
- prove de Moivre's theorem for a positive integer exponent
- use de Moivre's theorem for a positive or negative rational exponent
  - to express trigonometrical ratios of multiple angles in terms of powers of trigonometrical ratios of the fundamental angle
  - to express powers of  $\sin \theta$  and  $\cos \theta$  in terms of multiple angles
  - in the summation of series
  - in finding and using the  $n$ th roots of unity.

#### Notes and examples

e.g. by induction.

e.g. expressing  $\cos 5\theta$  in terms of  $\cos \theta$  or  $\tan 5\theta$  in terms of  $\tan \theta$ .

e.g. expressing  $\sin^6 \theta$  in terms of  $\cos 2\theta$ ,  $\cos 4\theta$  and  $\cos 6\theta$ .

e.g. using the ' $C + iS$ ' method to sum series such

as  $\sum_{r=1}^n \binom{n}{r} \sin r\theta$ .

### 2.6 Differential equations

#### Candidates should be able to:

- find an integrating factor for a first order linear differential equation, and use an integrating factor to find the general solution
- recall the meaning of the terms 'complementary function' and 'particular integral' in the context of linear differential equations, and recall that the general solution is the sum of the complementary function and a particular integral
- find the complementary function for a first or second order linear differential equation with constant coefficients
- recall the form of, and find, a particular integral for a first or second order linear differential equation in the cases where a polynomial or  $a e^{bx}$  or  $a \cos px + b \sin px$  is a suitable form, and in other simple cases find the appropriate coefficient(s) given a suitable form of particular integral

#### Notes and examples

e.g.  $\frac{dy}{dx} - 2y = x^2$ ,  $x \frac{dy}{dx} - y = x^4$ ,

$\frac{dy}{dx} + y \coth x = \cosh x$ .

For second order equations, including the cases where the auxiliary equation has distinct real roots, a repeated real root or conjugate complex roots.

e.g. evaluate  $k$  given that  $kx \cos 2x$  is a particular

integral of  $\frac{d^2y}{dx^2} + 4y = \sin 2x$ .

*continued*

## 2 Further Pure Mathematics 2

### 2.6 Differential equations continued

Candidates should be able to:

- use a given substitution to reduce a differential equation to a first or second order linear equation with constant coefficients or to a first order equation with separable variables
- use initial conditions to find a particular solution to a differential equation, and interpret a solution in terms of a problem modelled by a differential equation.

Notes and examples

e.g. the substitution  $x = e^t$  to reduce to linear form a differential equation with terms of the form

$$ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy, \text{ or the substitution } y = ux \text{ to}$$

reduce  $\frac{dy}{dx} = \frac{x+y}{x-y}$  to separable form.

### 3 Further Mechanics (for Paper 3)

Knowledge of Cambridge International AS & A Level Mathematics (9709) Paper 4: Mechanics subject content is assumed for this component.

#### 3.1 Motion of a projectile

Candidates should be able to:

- model the motion of a projectile as a particle moving with constant acceleration and understand any limitations of the model
- use horizontal and vertical equations of motion to solve problems on the motion of projectiles, including finding the magnitude and direction of the velocity at a given time or position, the range on a horizontal plane and the greatest height reached
- derive and use the Cartesian equation of the trajectory of a projectile, including problems in which the initial speed and/or angle of projection may be unknown.

Notes and examples

Vector methods are not required.

Knowledge of the 'bounding parabola' for accessible points is not included.

#### 3.2 Equilibrium of a rigid body

Candidates should be able to:

- calculate the moment of a force about a point
- use the result that the effect of gravity on a rigid body is equivalent to a single force acting at the centre of mass of the body, and identify the position of the centre of mass of a uniform body using considerations of symmetry
- use given information about the position of the centre of mass of a triangular lamina and other simple shapes
- determine the position of the centre of mass of a composite body by considering an equivalent system of particles
- use the principle that if a rigid body is in equilibrium under the action of coplanar forces then the vector sum of the forces is zero and the sum of the moments of the forces about any point is zero, and the converse of this
- solve problems involving the equilibrium of a single rigid body under the action of coplanar forces, including those involving toppling or sliding.

Notes and examples

For questions involving coplanar forces only; understanding of the vector nature of moments is not required.

Proofs of results given in the MF19 List of formulae are not required.

Simple cases only, e.g. a uniform L-shaped lamina, or a uniform cone joined at its base to a uniform hemisphere of the same radius.



### 3 Further Mechanics

#### 3.3 Circular motion

##### Candidates should be able to:

- understand the concept of angular speed for a particle moving in a circle, and use the relation  $v = r\omega$
- understand that the acceleration of a particle moving in a circle with constant speed is directed towards the centre of the circle, and use the formulae  $r\omega^2$  and  $\frac{v^2}{r}$ .
- solve problems which can be modelled by the motion of a particle moving in a horizontal circle with constant speed
- solve problems which can be modelled by the motion of a particle in a vertical circle without loss of energy.

##### Notes and examples

Proof of the acceleration formulae is not required.

Including finding a normal contact force or the tension in a string, locating points at which these are zero, and conditions for complete circular motion.

#### 3.4 Hooke's law

##### Candidates should be able to:

- use Hooke's law as a model relating the force in an elastic string or spring to the extension or compression, and understand the term modulus of elasticity
- use the formula for the elastic potential energy stored in a string or spring
- solve problems involving forces due to elastic strings or springs, including those where considerations of work and energy are needed.

##### Notes and examples

Proof of the formula is not required.

e.g. a particle moving horizontally or vertically or on an inclined plane while attached to one or more strings or springs, or a particle attached to an elastic string acting as a 'conical pendulum'.

#### 3.5 Linear motion under a variable force

##### Candidates should be able to:

- solve problems which can be modelled as the linear motion of a particle under the action of a variable force, by setting up and solving an appropriate differential equation.

##### Notes and examples

Including use of  $v \frac{dv}{dx}$  for acceleration, where appropriate.

Calculus required is restricted to content from Pure Mathematics 3 in Cambridge International A Level Mathematics (9709).

Only differential equations in which the variables are separable are included.

### 3 Further Mechanics

#### 3.6 Momentum

##### Candidates should be able to:

- recall Newton's experimental law and the definition of the coefficient of restitution, the property  $0 \leq e \leq 1$ , and the meaning of the terms 'perfectly elastic' ( $e = 1$ ) and 'inelastic' ( $e = 0$ )
- use conservation of linear momentum and/or Newton's experimental law to solve problems that may be modelled as the direct or oblique impact of two smooth spheres, or the direct or oblique impact of a smooth sphere with a fixed surface.

##### Notes and examples

## 4 Further Probability & Statistics (for Paper 4)

Knowledge of Cambridge International AS & A Level Mathematics (9709) Papers 5 and 6: Probability & Statistics subject content is assumed for this component.

Please see the support document *Guide to prior learning for Paper 4 Further Probability & Statistics* on the Cambridge website for recommended prior knowledge for this paper.

### 4.1 Continuous random variables

Candidates should be able to:

Notes and examples

- use a probability density function which may be defined piecewise
- use the general result  $E(g(X)) = \int f(x)g(x)dx$  where  $f(x)$  is the probability density function of the continuous random variable  $X$  and  $g(X)$  is a function of  $X$
- understand and use the relationship between the probability density function (PDF) and the cumulative distribution function (CDF), and use either to evaluate probabilities or percentiles
- use cumulative distribution functions (CDFs) of related variables in simple cases.

e.g. given the CDF of a variable  $X$ , find the CDF of a related variable  $Y$ , and hence its PDF, e.g. where  $Y = X^3$ .

### 4.2 Inference using normal and $t$ -distributions

Candidates should be able to:

Notes and examples

- formulate hypotheses and apply a hypothesis test concerning the population mean using a small sample drawn from a normal population of unknown variance, using a  $t$ -test
- calculate a pooled estimate of a population variance from two samples
- formulate hypotheses concerning the difference of population means, and apply, as appropriate
  - a 2-sample  $t$ -test
  - a paired sample  $t$ -test
  - a test using a normal distribution
- determine a confidence interval for a population mean, based on a small sample from a normal population with unknown variance, using a  $t$ -distribution
- determine a confidence interval for a difference of population means, using a  $t$ -distribution or a normal distribution, as appropriate.

Calculations based on either raw or summarised data may be required.

The ability to select the test appropriate to the circumstances of a problem is expected.

## 4 Further Probability & Statistics

### 4.3 $\chi^2$ -tests

#### Candidates should be able to:

- fit a theoretical distribution, as prescribed by a given hypothesis, to given data
- use a  $\chi^2$ -test, with the appropriate number of degrees of freedom, to carry out the corresponding goodness of fit analysis
- use a  $\chi^2$ -test, with the appropriate number of degrees of freedom, for independence in a contingency table.

#### Notes and examples

Questions will not involve lengthy calculations.

Classes should be combined so that each expected frequency is at least 5.

Yates' correction is not required.

Where appropriate, either rows or columns should be combined so that the expected frequency in each cell is at least 5.

### 4.4 Non-parametric tests

#### Candidates should be able to:

- understand the idea of a non-parametric test and appreciate situations in which such a test might be useful
- understand the basis of the sign test, the Wilcoxon signed-rank test and the Wilcoxon rank-sum test
- use a single-sample sign test and a single-sample Wilcoxon signed-rank test to test a hypothesis concerning a population median
- use a paired-sample sign test, a Wilcoxon matched-pairs signed-rank test and a Wilcoxon rank-sum test, as appropriate, to test for identity of populations.

#### Notes and examples

e.g. when sampling from a population which cannot be assumed to be normally distributed.

Including knowledge that Wilcoxon tests are valid only for symmetrical distributions.

Including the use of normal approximations where appropriate.

Questions will not involve tied ranks or observations equal to the population median value being tested.

Including the use of normal approximations where appropriate.

Questions will not involve tied ranks or zero-difference pairs.

### 4.5 Probability generating functions

#### Candidates should be able to:

- understand the concept of a probability generating function (PGF) and construct and use the PGF for given distributions
- use formulae for the mean and variance of a discrete random variable in terms of its PGF, and use these formulae to calculate the mean and variance of a given probability distribution
- use the result that the PGF of the sum of independent random variables is the product of the PGFs of those random variables.

#### Notes and examples

Including the discrete uniform, binomial, geometric and Poisson distributions.

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## 4 Details of the assessment

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### Planning for assessment of AS & A Level Further Mathematics (9231)

AS & A Level Further Mathematics (9231) can be studied with AS & A Level Mathematics (9709). However, learners may have completed their course in AS & A Level Mathematics before starting the Further Mathematics course.

The AS Level syllabuses are not designed for taking AS Mathematics (9709) and AS Further Mathematics (9231) in parallel after only one year of study. This is because AS Further Mathematics (9231) topics may depend on assumed prior knowledge from A Level Mathematics (9709), such as the Paper 3: Pure Mathematics 3 subject content. Therefore you must plan your teaching carefully if you intend to enter candidates for AS Level Further Mathematics after one year.

See the introduction to section 3 of this syllabus for a summary of the assumed prior knowledge required for AS & A Level Further Mathematics (9231).

### Examination information

All components are assessed by written examinations which are externally marked. Sample assessment materials are available on our website at [www.cambridgeinternational.org](http://www.cambridgeinternational.org) showing the question style and level of the examination papers.

#### Application of mathematical techniques

As well as demonstrating the appropriate techniques, candidates need to apply their knowledge in solving problems. Individual examination questions may involve ideas and methods from more than one section of the syllabus content for that component.

The main focus of examination questions will be the AS & A Level Further Mathematics syllabus content. However, candidates may need to make use of prior knowledge and mathematical techniques from previous study, as listed in the introduction to section 3 of this syllabus.

#### Structure of the question paper

All questions in the examination papers are compulsory. An approximate number of questions for each paper is given in the Assessment overview in section 2 of this syllabus. Questions are of varied lengths and often contain several parts, labelled (a), (b), (c), which may have sub-parts (i), (ii), (iii), as needed. Some questions might require candidates to sketch graphs or diagrams, or draw accurate graphs.

#### Answer space

Candidates answer on the question paper. All working should be shown neatly and clearly in the spaces provided for each question. New questions often start on a fresh page, so more answer space may be provided than is needed. If additional space is required, candidates should use the lined page at the end of the question paper, where the question number or numbers must be clearly shown.

## Degrees of accuracy

Candidates should give non-exact numerical answers correct to three significant figures (or one decimal place for angles in degrees) unless a different level of accuracy is specified in the question. To earn accuracy marks, candidates should avoid rounding figures until they have their final answer.

## Additional materials for examinations

Candidates are expected to have the following equipment in examinations:

- a ruler
- a scientific calculator (see the following section).

Note: a protractor and a pair of compasses are not required.

A list of formulae and statistical tables (MF19) is supplied in examinations for the use of candidates. A copy of the list of formulae and tables is given for reference in section 5 of this syllabus. Note that MF19 is a combined formulae list for AS & A Level Mathematics (9709) and AS & A Level Further Mathematics (9231). All formulae in the list may be needed for this syllabus.

## Calculators

It is expected that candidates will have a calculator with standard 'scientific' functions available for use in all the examinations. Computers, graphical calculators and calculators capable of symbolic algebraic manipulation or symbolic differentiation or integration are not permitted. The General Regulations concerning the use of calculators are contained in the *Cambridge Handbook* at [www.cambridgeinternational.org/examsOfficers](http://www.cambridgeinternational.org/examsOfficers)

Candidates are expected to show all necessary working; no marks will be given for unsupported answers from a calculator.

## Mathematical notation

The list of mathematical notation that may be used in examinations for this syllabus is available on our website at [www.cambridgeinternational.org/9231](http://www.cambridgeinternational.org/9231)

## Command words

The table below includes command words used in the assessment for this syllabus. The use of the command word will relate to the subject context.

| Command word       | What it means   |
|--------------------|---|
| <b>Calculate</b>   | work out from given facts, figures or information   |
| <b>Deduce</b>      | conclude from available information   |
| <b>Derive</b>      | obtain something (expression/equation/value) from another by a sequence of logical steps  |
| <b>Describe</b>    | state the points of a topic / give characteristics and main features  |
| <b>Determine</b>   | establish with certainty  |
| <b>Evaluate</b>    | judge or calculate the quality, importance, amount, or value of something   |
| <b>Explain</b>     | set out purposes or reasons / make the relationships between things evident / provide why and/or how and support with relevant evidence |
| <b>Identify</b>    | name/select/recognise   |
| <b>Interpret</b>   | identify meaning or significance in relation to the context   |
| <b>Justify</b>     | support a case with evidence/argument   |
| <b>Prove</b>       | confirm the truth of the given statement using a chain of logical mathematical reasoning  |
| <b>Show (that)</b> | provide structured evidence that leads to a given result  |
| <b>Sketch</b>      | make a simple freehand drawing showing the key features   |
| <b>State</b>       | express in clear terms  |
| <b>Verify</b>      | confirm a given statement/result is true  |

## 5 List of formulae and statistical tables (MF19)

### PURE MATHEMATICS

#### Mensuration

$$\text{Volume of sphere} = \frac{4}{3}\pi r^3$$

$$\text{Surface area of sphere} = 4\pi r^2$$

$$\text{Volume of cone or pyramid} = \frac{1}{3} \times \text{base area} \times \text{height}$$

$$\text{Area of curved surface of cone} = \pi r \times \text{slant height}$$

$$\text{Arc length of circle} = r\theta \quad (\theta \text{ in radians})$$

$$\text{Area of sector of circle} = \frac{1}{2}r^2\theta \quad (\theta \text{ in radians})$$

#### Algebra

For the quadratic equation  $ax^2 + bx + c = 0$ :

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For an arithmetic series:

$$u_n = a + (n-1)d, \quad S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n\{2a + (n-1)d\}$$

For a geometric series:

$$u_n = ar^{n-1}, \quad S_n = \frac{a(1-r^n)}{1-r} \quad (r \neq 1), \quad S_\infty = \frac{a}{1-r} \quad (|r| < 1)$$

Binomial series:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \dots + b^n, \text{ where } n \text{ is a positive integer}$$

$$\text{and } \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots, \text{ where } n \text{ is rational and } |x| < 1$$



*Trigonometry*

$$\tan \theta \equiv \frac{\sin \theta}{\cos \theta}$$

$$\cos^2 \theta + \sin^2 \theta \equiv 1, \quad 1 + \tan^2 \theta \equiv \sec^2 \theta, \quad \cot^2 \theta + 1 \equiv \operatorname{cosec}^2 \theta$$

$$\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A \equiv 2 \sin A \cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2 \cos^2 A - 1 \equiv 1 - 2 \sin^2 A$$

$$\tan 2A \equiv \frac{2 \tan A}{1 - \tan^2 A}$$

Principal values:

$$-\frac{1}{2}\pi \leq \sin^{-1} x \leq \frac{1}{2}\pi, \quad 0 \leq \cos^{-1} x \leq \pi, \quad -\frac{1}{2}\pi < \tan^{-1} x < \frac{1}{2}\pi$$

*Differentiation*

| <b>f(x)</b>              | <b>f'(x)</b>                                    |
|--------------------------|---|
| $x^n$                    | $nx^{n-1}$                                      |
| $\ln x$                  | $\frac{1}{x}$                                   |
| $e^x$                    | $e^x$   |
| $\sin x$                 | $\cos x$  |
| $\cos x$                 | $-\sin x$                                       |
| $\tan x$                 | $\sec^2 x$                                      |
| $\sec x$                 | $\sec x \tan x$                                 |
| $\operatorname{cosec} x$ | $-\operatorname{cosec} x \cot x$                |
| $\cot x$                 | $-\operatorname{cosec}^2 x$                     |
| $\tan^{-1} x$            | $\frac{1}{1+x^2}$                               |
| $uv$                     | $v \frac{du}{dx} + u \frac{dv}{dx}$             |
| $\frac{u}{v}$            | $\frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$ |

If  $x = f(t)$  and  $y = g(t)$  then  $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

*Integration*(Arbitrary constants are omitted;  $a$  denotes a positive constant.)

| $f(x)$                | $\int f(x) dx$                                  |               |
|-----------------------|---|---------------|
| $x^n$                 | $\frac{x^{n+1}}{n+1}$                           | $(n \neq -1)$ |
| $\frac{1}{x}$         | $\ln x $  |               |
| $e^x$                 | $e^x$   |               |
| $\sin x$              | $-\cos x$                                       |               |
| $\cos x$              | $\sin x$  |               |
| $\sec^2 x$            | $\tan x$  |               |
| $\frac{1}{x^2 + a^2}$ | $\frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$ |               |
| $\frac{1}{x^2 - a^2}$ | $\frac{1}{2a} \ln\left \frac{x-a}{x+a}\right $  | $(x > a)$     |
| $\frac{1}{a^2 - x^2}$ | $\frac{1}{2a} \ln\left \frac{a+x}{a-x}\right $  | $( x  < a)$   |

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)|$$

*Vectors*If  $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$  and  $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$  then

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

**FURTHER PURE MATHEMATICS***Algebra*

Summations:

$$\sum_{r=1}^n r = \frac{1}{2}n(n+1), \quad \sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1), \quad \sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2$$

Maclaurin's series:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^r}{r!} f^{(r)}(0) + \dots$$

$$e^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^r}{r!} + \dots \quad (\text{all } x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{r+1} \frac{x^r}{r} + \dots \quad (-1 < x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \dots \quad (\text{all } x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^r \frac{x^{2r}}{(2r)!} + \dots \quad (\text{all } x)$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^r \frac{x^{2r+1}}{2r+1} + \dots \quad (-1 \leq x \leq 1)$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2r+1}}{(2r+1)!} + \dots \quad (\text{all } x)$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2r}}{(2r)!} + \dots \quad (\text{all } x)$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots + \frac{x^{2r+1}}{2r+1} + \dots \quad (-1 < x < 1)$$

*Trigonometry*If  $t = \tan \frac{1}{2}x$  then:

$$\sin x = \frac{2t}{1+t^2} \quad \text{and} \quad \cos x = \frac{1-t^2}{1+t^2}$$

*Hyperbolic functions*

$$\cosh^2 x - \sinh^2 x \equiv 1, \quad \sinh 2x \equiv 2 \sinh x \cosh x, \quad \cosh 2x \equiv \cosh^2 x + \sinh^2 x$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \quad (x \geq 1)$$

$$\tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) \quad (|x| < 1)$$

*Differentiation*

| $f(x)$         | $f'(x)$                   |
|----------------|---------------------------|
| $\sin^{-1} x$  | $\frac{1}{\sqrt{1-x^2}}$  |
| $\cos^{-1} x$  | $-\frac{1}{\sqrt{1-x^2}}$ |
| $\sinh x$      | $\cosh x$                 |
| $\cosh x$      | $\sinh x$                 |
| $\tanh x$      | $\operatorname{sech}^2 x$ |
| $\sinh^{-1} x$ | $\frac{1}{\sqrt{1+x^2}}$  |
| $\cosh^{-1} x$ | $\frac{1}{\sqrt{x^2-1}}$  |
| $\tanh^{-1} x$ | $\frac{1}{1-x^2}$         |

*Integration*(Arbitrary constants are omitted;  $a$  denotes a positive constant.)

| $f(x)$                     | $\int f(x) dx$  |                          |
|----------------------------|---|--------------------------|
| $\sec x$                   | $\ln  \sec x + \tan x  = \ln  \tan(\frac{1}{2}x + \frac{1}{4}\pi) $ | $( x  < \frac{1}{2}\pi)$ |
| $\operatorname{cosec} x$   | $-\ln  \operatorname{cosec} x + \cot x  = \ln  \tan(\frac{1}{2}x) $ | $(0 < x < \pi)$          |
| $\sinh x$                  | $\cosh x$   |                          |
| $\cosh x$                  | $\sinh x$   |                          |
| $\operatorname{sech}^2 x$  | $\tanh x$   |                          |
| $\frac{1}{\sqrt{a^2-x^2}}$ | $\sin^{-1}\left(\frac{x}{a}\right)$                                 | $( x  < a)$              |
| $\frac{1}{\sqrt{x^2-a^2}}$ | $\cosh^{-1}\left(\frac{x}{a}\right)$                                | $(x > a)$                |
| $\frac{1}{\sqrt{a^2+x^2}}$ | $\sinh^{-1}\left(\frac{x}{a}\right)$                                |                          |

**MECHANICS***Uniformly accelerated motion*

$$v = u + at, \quad s = \frac{1}{2}(u + v)t, \quad s = ut + \frac{1}{2}at^2, \quad v^2 = u^2 + 2as$$

**FURTHER MECHANICS***Motion of a projectile*

Equation of trajectory is:

$$y = x \tan \theta - \frac{gx^2}{2V^2 \cos^2 \theta}$$

*Elastic strings and springs*

$$T = \frac{\lambda x}{l}, \quad E = \frac{\lambda x^2}{2l}$$

*Motion in a circle*

For uniform circular motion, the acceleration is directed towards the centre and has magnitude

$$\omega^2 r \quad \text{or} \quad \frac{v^2}{r}$$

*Centres of mass of uniform bodies*Triangular lamina:  $\frac{2}{3}$  along median from vertexSolid hemisphere of radius  $r$ :  $\frac{3}{8}r$  from centreHemispherical shell of radius  $r$ :  $\frac{1}{2}r$  from centreCircular arc of radius  $r$  and angle  $2\alpha$ :  $\frac{r \sin \alpha}{\alpha}$  from centreCircular sector of radius  $r$  and angle  $2\alpha$ :  $\frac{2r \sin \alpha}{3\alpha}$  from centreSolid cone or pyramid of height  $h$ :  $\frac{3}{4}h$  from vertex

## PROBABILITY & STATISTICS

### Summary statistics

For ungrouped data:

$$\bar{x} = \frac{\Sigma x}{n}, \quad \text{standard deviation} = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n}} = \sqrt{\frac{\Sigma x^2}{n} - \bar{x}^2}$$

For grouped data:

$$\bar{x} = \frac{\Sigma xf}{\Sigma f}, \quad \text{standard deviation} = \sqrt{\frac{\Sigma(x - \bar{x})^2 f}{\Sigma f}} = \sqrt{\frac{\Sigma x^2 f}{\Sigma f} - \bar{x}^2}$$

### Discrete random variables

$$E(X) = \Sigma xp, \quad \text{Var}(X) = \Sigma x^2 p - \{E(X)\}^2$$

For the binomial distribution  $B(n, p)$ :

$$p_r = \binom{n}{r} p^r (1-p)^{n-r}, \quad \mu = np, \quad \sigma^2 = np(1-p)$$

For the geometric distribution  $\text{Geo}(p)$ :

$$p_r = p(1-p)^{r-1}, \quad \mu = \frac{1}{p}$$

For the Poisson distribution  $\text{Po}(\lambda)$

$$p_r = e^{-\lambda} \frac{\lambda^r}{r!}, \quad \mu = \lambda, \quad \sigma^2 = \lambda$$

### Continuous random variables

$$E(X) = \int x f(x) dx, \quad \text{Var}(X) = \int x^2 f(x) dx - \{E(X)\}^2$$

### Sampling and testing

Unbiased estimators:

$$\bar{x} = \frac{\Sigma x}{n}, \quad s^2 = \frac{\Sigma(x - \bar{x})^2}{n-1} = \frac{1}{n-1} \left( \Sigma x^2 - \frac{(\Sigma x)^2}{n} \right)$$

Central Limit Theorem:

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Approximate distribution of sample proportion:

$$N\left(p, \frac{p(1-p)}{n}\right)$$

**FURTHER PROBABILITY & STATISTICS***Sampling and testing*

Two-sample estimate of a common variance:

$$s^2 = \frac{\Sigma(x_1 - \bar{x}_1)^2 + \Sigma(x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}$$

*Probability generating functions*

$$G_X(t) = E(t^X),$$

$$E(X) = G'_X(1),$$

$$\text{Var}(X) = G''_X(1) + G'_X(1) - \{G'_X(1)\}^2$$

**THE NORMAL DISTRIBUTION FUNCTION**

If  $Z$  has a normal distribution with mean 0 and variance 1, then, for each value of  $z$ , the table gives the value of  $\Phi(z)$ , where

$$\Phi(z) = P(Z \leq z).$$



For negative values of  $z$ , use  $\Phi(-z) = 1 - \Phi(z)$ .

| $z$ |        |        |        |        |        |        |        |        |        |        | ADD |   |    |    |    |    |    |    |    |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|-----|---|----|----|----|----|----|----|----|
|     | 0      | 1      | 2      | 3      | 4      | 5      | 6      | 7      | 8      | 9      | 1   | 2 | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 | 4   | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 | 4   | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 | 4   | 8 | 12 | 15 | 19 | 23 | 27 | 31 | 35 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 | 4   | 7 | 11 | 15 | 19 | 22 | 26 | 30 | 34 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 | 4   | 7 | 11 | 14 | 18 | 22 | 25 | 29 | 32 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 | 3   | 7 | 10 | 14 | 17 | 20 | 24 | 27 | 31 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 | 3   | 7 | 10 | 13 | 16 | 19 | 23 | 26 | 29 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 | 3   | 6 | 9  | 12 | 15 | 18 | 21 | 24 | 27 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 | 3   | 5 | 8  | 11 | 14 | 16 | 19 | 22 | 25 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 | 3   | 5 | 8  | 10 | 13 | 15 | 18 | 20 | 23 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 | 2   | 5 | 7  | 9  | 12 | 14 | 16 | 19 | 21 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 | 2   | 4 | 6  | 8  | 10 | 12 | 14 | 16 | 18 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 | 2   | 4 | 6  | 7  | 9  | 11 | 13 | 15 | 17 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 | 2   | 3 | 5  | 6  | 8  | 10 | 11 | 13 | 14 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 | 1   | 3 | 4  | 6  | 7  | 8  | 10 | 11 | 13 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 | 1   | 2 | 4  | 5  | 6  | 7  | 8  | 10 | 11 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 | 1   | 2 | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 | 1   | 2 | 3  | 4  | 4  | 5  | 6  | 7  | 8  |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 | 1   | 1 | 2  | 3  | 4  | 4  | 5  | 6  | 6  |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 | 1   | 1 | 2  | 2  | 3  | 4  | 4  | 5  | 5  |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 | 0   | 1 | 1  | 2  | 2  | 3  | 3  | 4  | 4  |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 | 0   | 1 | 1  | 2  | 2  | 2  | 3  | 3  | 4  |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 | 0   | 1 | 1  | 1  | 2  | 2  | 2  | 3  | 3  |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 | 0   | 1 | 1  | 1  | 1  | 2  | 2  | 2  | 2  |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 | 0   | 0 | 1  | 1  | 1  | 1  | 1  | 2  | 2  |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 | 0   | 0 | 0  | 1  | 1  | 1  | 1  | 1  | 1  |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 | 0   | 0 | 0  | 0  | 1  | 1  | 1  | 1  | 1  |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 | 0   | 0 | 0  | 0  | 0  | 1  | 1  | 1  | 1  |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 | 0   | 0 | 0  | 0  | 0  | 0  | 0  | 1  | 1  |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 | 0   | 0 | 0  | 0  | 0  | 0  | 0  | 0  | 0  |

**Critical values for the normal distribution**

If  $Z$  has a normal distribution with mean 0 and variance 1, then, for each value of  $p$ , the table gives the value of  $z$  such that

$$P(Z \leq z) = p.$$

|     |       |       |       |       |       |       |        |       |        |
|-----|-------|-------|-------|-------|-------|-------|--------|-------|--------|
| $p$ | 0.75  | 0.90  | 0.95  | 0.975 | 0.99  | 0.995 | 0.9975 | 0.999 | 0.9995 |
| $z$ | 0.674 | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 | 2.807  | 3.090 | 3.291  |



### CRITICAL VALUES FOR THE $t$ -DISTRIBUTION

If  $T$  has a  $t$ -distribution with  $\nu$  degrees of freedom, then, for each pair of values of  $p$  and  $\nu$ , the table gives the value of  $t$  such that:

$$P(T \leq t) = p.$$



| $p$      | 0.75  | 0.90  | 0.95  | 0.975 | 0.99  | 0.995 | 0.9975 | 0.999 | 0.9995 |
|----------|-------|-------|-------|-------|-------|-------|--------|-------|--------|
| $\nu=1$  | 1.000 | 3.078 | 6.314 | 12.71 | 31.82 | 63.66 | 127.3  | 318.3 | 636.6  |
| 2        | 0.816 | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 | 14.09  | 22.33 | 31.60  |
| 3        | 0.765 | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 | 7.453  | 10.21 | 12.92  |
| 4        | 0.741 | 1.533 | 2.132 | 2.776 | 3.747 | 4.604 | 5.598  | 7.173 | 8.610  |
| 5        | 0.727 | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 | 4.773  | 5.894 | 6.869  |
| 6        | 0.718 | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 | 4.317  | 5.208 | 5.959  |
| 7        | 0.711 | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 | 4.029  | 4.785 | 5.408  |
| 8        | 0.706 | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 | 3.833  | 4.501 | 5.041  |
| 9        | 0.703 | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 | 3.690  | 4.297 | 4.781  |
| 10       | 0.700 | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 | 3.581  | 4.144 | 4.587  |
| 11       | 0.697 | 1.363 | 1.796 | 2.201 | 2.718 | 3.106 | 3.497  | 4.025 | 4.437  |
| 12       | 0.695 | 1.356 | 1.782 | 2.179 | 2.681 | 3.055 | 3.428  | 3.930 | 4.318  |
| 13       | 0.694 | 1.350 | 1.771 | 2.160 | 2.650 | 3.012 | 3.372  | 3.852 | 4.221  |
| 14       | 0.692 | 1.345 | 1.761 | 2.145 | 2.624 | 2.977 | 3.326  | 3.787 | 4.140  |
| 15       | 0.691 | 1.341 | 1.753 | 2.131 | 2.602 | 2.947 | 3.286  | 3.733 | 4.073  |
| 16       | 0.690 | 1.337 | 1.746 | 2.120 | 2.583 | 2.921 | 3.252  | 3.686 | 4.015  |
| 17       | 0.689 | 1.333 | 1.740 | 2.110 | 2.567 | 2.898 | 3.222  | 3.646 | 3.965  |
| 18       | 0.688 | 1.330 | 1.734 | 2.101 | 2.552 | 2.878 | 3.197  | 3.610 | 3.922  |
| 19       | 0.688 | 1.328 | 1.729 | 2.093 | 2.539 | 2.861 | 3.174  | 3.579 | 3.883  |
| 20       | 0.687 | 1.325 | 1.725 | 2.086 | 2.528 | 2.845 | 3.153  | 3.552 | 3.850  |
| 21       | 0.686 | 1.323 | 1.721 | 2.080 | 2.518 | 2.831 | 3.135  | 3.527 | 3.819  |
| 22       | 0.686 | 1.321 | 1.717 | 2.074 | 2.508 | 2.819 | 3.119  | 3.505 | 3.792  |
| 23       | 0.685 | 1.319 | 1.714 | 2.069 | 2.500 | 2.807 | 3.104  | 3.485 | 3.768  |
| 24       | 0.685 | 1.318 | 1.711 | 2.064 | 2.492 | 2.797 | 3.091  | 3.467 | 3.745  |
| 25       | 0.684 | 1.316 | 1.708 | 2.060 | 2.485 | 2.787 | 3.078  | 3.450 | 3.725  |
| 26       | 0.684 | 1.315 | 1.706 | 2.056 | 2.479 | 2.779 | 3.067  | 3.435 | 3.707  |
| 27       | 0.684 | 1.314 | 1.703 | 2.052 | 2.473 | 2.771 | 3.057  | 3.421 | 3.689  |
| 28       | 0.683 | 1.313 | 1.701 | 2.048 | 2.467 | 2.763 | 3.047  | 3.408 | 3.674  |
| 29       | 0.683 | 1.311 | 1.699 | 2.045 | 2.462 | 2.756 | 3.038  | 3.396 | 3.660  |
| 30       | 0.683 | 1.310 | 1.697 | 2.042 | 2.457 | 2.750 | 3.030  | 3.385 | 3.646  |
| 40       | 0.681 | 1.303 | 1.684 | 2.021 | 2.423 | 2.704 | 2.971  | 3.307 | 3.551  |
| 60       | 0.679 | 1.296 | 1.671 | 2.000 | 2.390 | 2.660 | 2.915  | 3.232 | 3.460  |
| 120      | 0.677 | 1.289 | 1.658 | 1.980 | 2.358 | 2.617 | 2.860  | 3.160 | 3.373  |
| $\infty$ | 0.674 | 1.282 | 1.645 | 1.960 | 2.326 | 2.576 | 2.807  | 3.090 | 3.291  |

### CRITICAL VALUES FOR THE $\chi^2$ -DISTRIBUTION

If  $X$  has a  $\chi^2$ -distribution with  $\nu$  degrees of freedom then, for each pair of values of  $p$  and  $\nu$ , the table gives the value of  $x$  such that

$$P(X \leq x) = p.$$



| $p$     | 0.01                  | 0.025                 | 0.05                  | 0.9   | 0.95  | 0.975 | 0.99  | 0.995 | 0.999 |
|---------|-----------------------|-----------------------|-----------------------|-------|-------|-------|-------|-------|-------|
| $\nu=1$ | 0.0 <sup>3</sup> 1571 | 0.0 <sup>3</sup> 9821 | 0.0 <sup>2</sup> 3932 | 2.706 | 3.841 | 5.024 | 6.635 | 7.879 | 10.83 |
| 2       | 0.02010               | 0.05064               | 0.1026                | 4.605 | 5.991 | 7.378 | 9.210 | 10.60 | 13.82 |
| 3       | 0.1148                | 0.2158                | 0.3518                | 6.251 | 7.815 | 9.348 | 11.34 | 12.84 | 16.27 |
| 4       | 0.2971                | 0.4844                | 0.7107                | 7.779 | 9.488 | 11.14 | 13.28 | 14.86 | 18.47 |
| 5       | 0.5543                | 0.8312                | 1.145                 | 9.236 | 11.07 | 12.83 | 15.09 | 16.75 | 20.51 |
| 6       | 0.8721                | 1.237                 | 1.635                 | 10.64 | 12.59 | 14.45 | 16.81 | 18.55 | 22.46 |
| 7       | 1.239                 | 1.690                 | 2.167                 | 12.02 | 14.07 | 16.01 | 18.48 | 20.28 | 24.32 |
| 8       | 1.647                 | 2.180                 | 2.733                 | 13.36 | 15.51 | 17.53 | 20.09 | 21.95 | 26.12 |
| 9       | 2.088                 | 2.700                 | 3.325                 | 14.68 | 16.92 | 19.02 | 21.67 | 23.59 | 27.88 |
| 10      | 2.558                 | 3.247                 | 3.940                 | 15.99 | 18.31 | 20.48 | 23.21 | 25.19 | 29.59 |
| 11      | 3.053                 | 3.816                 | 4.575                 | 17.28 | 19.68 | 21.92 | 24.73 | 26.76 | 31.26 |
| 12      | 3.571                 | 4.404                 | 5.226                 | 18.55 | 21.03 | 23.34 | 26.22 | 28.30 | 32.91 |
| 13      | 4.107                 | 5.009                 | 5.892                 | 19.81 | 22.36 | 24.74 | 27.69 | 29.82 | 34.53 |
| 14      | 4.660                 | 5.629                 | 6.571                 | 21.06 | 23.68 | 26.12 | 29.14 | 31.32 | 36.12 |
| 15      | 5.229                 | 6.262                 | 7.261                 | 22.31 | 25.00 | 27.49 | 30.58 | 32.80 | 37.70 |
| 16      | 5.812                 | 6.908                 | 7.962                 | 23.54 | 26.30 | 28.85 | 32.00 | 34.27 | 39.25 |
| 17      | 6.408                 | 7.564                 | 8.672                 | 24.77 | 27.59 | 30.19 | 33.41 | 35.72 | 40.79 |
| 18      | 7.015                 | 8.231                 | 9.390                 | 25.99 | 28.87 | 31.53 | 34.81 | 37.16 | 42.31 |
| 19      | 7.633                 | 8.907                 | 10.12                 | 27.20 | 30.14 | 32.85 | 36.19 | 38.58 | 43.82 |
| 20      | 8.260                 | 9.591                 | 10.85                 | 28.41 | 31.41 | 34.17 | 37.57 | 40.00 | 45.31 |
| 21      | 8.897                 | 10.28                 | 11.59                 | 29.62 | 32.67 | 35.48 | 38.93 | 41.40 | 46.80 |
| 22      | 9.542                 | 10.98                 | 12.34                 | 30.81 | 33.92 | 36.78 | 40.29 | 42.80 | 48.27 |
| 23      | 10.20                 | 11.69                 | 13.09                 | 32.01 | 35.17 | 38.08 | 41.64 | 44.18 | 49.73 |
| 24      | 10.86                 | 12.40                 | 13.85                 | 33.20 | 36.42 | 39.36 | 42.98 | 45.56 | 51.18 |
| 25      | 11.52                 | 13.12                 | 14.61                 | 34.38 | 37.65 | 40.65 | 44.31 | 46.93 | 52.62 |
| 30      | 14.95                 | 16.79                 | 18.49                 | 40.26 | 43.77 | 46.98 | 50.89 | 53.67 | 59.70 |
| 40      | 22.16                 | 24.43                 | 26.51                 | 51.81 | 55.76 | 59.34 | 63.69 | 66.77 | 73.40 |
| 50      | 29.71                 | 32.36                 | 34.76                 | 63.17 | 67.50 | 71.42 | 76.15 | 79.49 | 86.66 |
| 60      | 37.48                 | 40.48                 | 43.19                 | 74.40 | 79.08 | 83.30 | 88.38 | 91.95 | 99.61 |
| 70      | 45.44                 | 48.76                 | 51.74                 | 85.53 | 90.53 | 95.02 | 100.4 | 104.2 | 112.3 |
| 80      | 53.54                 | 57.15                 | 60.39                 | 96.58 | 101.9 | 106.6 | 112.3 | 116.3 | 124.8 |
| 90      | 61.75                 | 65.65                 | 69.13                 | 107.6 | 113.1 | 118.1 | 124.1 | 128.3 | 137.2 |
| 100     | 70.06                 | 74.22                 | 77.93                 | 118.5 | 124.3 | 129.6 | 135.8 | 140.2 | 149.4 |

**WILCOXON SIGNED-RANK TEST**

The sample has size  $n$ .

$P$  is the sum of the ranks corresponding to the positive differences.

$Q$  is the sum of the ranks corresponding to the negative differences.

$T$  is the smaller of  $P$  and  $Q$ .

For each value of  $n$  the table gives the **largest** value of  $T$  which will lead to rejection of the null hypothesis at the level of significance indicated.

**Critical values of  $T$** 

|            | Level of significance |       |      |       |
|------------|-----------------------|-------|------|-------|
|            | 0.05                  | 0.025 | 0.01 | 0.005 |
| One-tailed | 0.05                  | 0.025 | 0.01 | 0.005 |
| Two-tailed | 0.1                   | 0.05  | 0.02 | 0.01  |
| $n = 6$    | 2                     | 0     |      |       |
| 7          | 3                     | 2     | 0    |       |
| 8          | 5                     | 3     | 1    | 0     |
| 9          | 8                     | 5     | 3    | 1     |
| 10         | 10                    | 8     | 5    | 3     |
| 11         | 13                    | 10    | 7    | 5     |
| 12         | 17                    | 13    | 9    | 7     |
| 13         | 21                    | 17    | 12   | 9     |
| 14         | 25                    | 21    | 15   | 12    |
| 15         | 30                    | 25    | 19   | 15    |
| 16         | 35                    | 29    | 23   | 19    |
| 17         | 41                    | 34    | 27   | 23    |
| 18         | 47                    | 40    | 32   | 27    |
| 19         | 53                    | 46    | 37   | 32    |
| 20         | 60                    | 52    | 43   | 37    |

For larger values of  $n$ , each of  $P$  and  $Q$  can be approximated by the normal distribution with mean  $\frac{1}{4}n(n+1)$  and variance  $\frac{1}{24}n(n+1)(2n+1)$ .

### WILCOXON RANK-SUM TEST

The two samples have sizes  $m$  and  $n$ , where  $m \leq n$ .

$R_m$  is the sum of the ranks of the items in the sample of size  $m$ .

$W$  is the smaller of  $R_m$  and  $m(n + m + 1) - R_m$ .

For each pair of values of  $m$  and  $n$ , the table gives the **largest** value of  $W$  which will lead to rejection of the null hypothesis at the level of significance indicated.

#### Critical values of $W$

|            | Level of significance |       |      |         |       |      |         |       |      |         |       |      |
|------------|-----------------------|-------|------|---------|-------|------|---------|-------|------|---------|-------|------|
|            | 0.05                  | 0.025 | 0.01 | 0.05    | 0.025 | 0.01 | 0.05    | 0.025 | 0.01 | 0.05    | 0.025 | 0.01 |
| One-tailed | 0.05                  | 0.025 | 0.01 | 0.05    | 0.025 | 0.01 | 0.05    | 0.025 | 0.01 | 0.05    | 0.025 | 0.01 |
| Two-tailed | 0.1                   | 0.05  | 0.02 | 0.1     | 0.05  | 0.02 | 0.1     | 0.05  | 0.02 | 0.1     | 0.05  | 0.02 |
| $n$        | $m = 3$               |       |      | $m = 4$ |       |      | $m = 5$ |       |      | $m = 6$ |       |      |
| 3          | 6                     | –     | –    |         |       |      |         |       |      |         |       |      |
| 4          | 6                     | –     | –    | 11      | 10    | –    |         |       |      |         |       |      |
| 5          | 7                     | 6     | –    | 12      | 11    | 10   | 19      | 17    | 16   |         |       |      |
| 6          | 8                     | 7     | –    | 13      | 12    | 11   | 20      | 18    | 17   | 28      | 26    | 24   |
| 7          | 8                     | 7     | 6    | 14      | 13    | 11   | 21      | 20    | 18   | 29      | 27    | 25   |
| 8          | 9                     | 8     | 6    | 15      | 14    | 12   | 23      | 21    | 19   | 31      | 29    | 27   |
| 9          | 10                    | 8     | 7    | 16      | 14    | 13   | 24      | 22    | 20   | 33      | 31    | 28   |
| 10         | 10                    | 9     | 7    | 17      | 15    | 13   | 26      | 23    | 21   | 35      | 32    | 29   |

|            | Level of significance |       |      |         |       |      |         |       |      |          |       |      |
|------------|-----------------------|-------|------|---------|-------|------|---------|-------|------|----------|-------|------|
|            | 0.05                  | 0.025 | 0.01 | 0.05    | 0.025 | 0.01 | 0.05    | 0.025 | 0.01 | 0.05     | 0.025 | 0.01 |
| One-tailed | 0.05                  | 0.025 | 0.01 | 0.05    | 0.025 | 0.01 | 0.05    | 0.025 | 0.01 | 0.05     | 0.025 | 0.01 |
| Two-tailed | 0.1                   | 0.05  | 0.02 | 0.1     | 0.05  | 0.02 | 0.1     | 0.05  | 0.02 | 0.1      | 0.05  | 0.02 |
| $n$        | $m = 7$               |       |      | $m = 8$ |       |      | $m = 9$ |       |      | $m = 10$ |       |      |
| 7          | 39                    | 36    | 34   |         |       |      |         |       |      |          |       |      |
| 8          | 41                    | 38    | 35   | 51      | 49    | 45   |         |       |      |          |       |      |
| 9          | 43                    | 40    | 37   | 54      | 51    | 47   | 66      | 62    | 59   |          |       |      |
| 10         | 45                    | 42    | 39   | 56      | 53    | 49   | 69      | 65    | 61   | 82       | 78    | 74   |

For larger values of  $m$  and  $n$ , the normal distribution with mean  $\frac{1}{2}m(m + n + 1)$  and variance  $\frac{1}{12}mn(m + n + 1)$  should be used as an approximation to the distribution of  $R_m$ .

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## 6 What else you need to know

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This section is an overview of other information you need to know about this syllabus. It will help to share the administrative information with your exams officer so they know when you will need their support. Find more information about our administrative processes at [www.cambridgeinternational.org/eoguide](http://www.cambridgeinternational.org/eoguide)

### Before you start

#### Previous study

We strongly recommend that learners starting this course should have studied, or be studying, Cambridge International AS & A Level Mathematics (9709) or the equivalent. See the introduction to section 3 of this syllabus for more details of expected prior knowledge.

#### Guided learning hours

We design Cambridge International AS & A Level syllabuses based on learners having about 180 guided learning hours for each Cambridge International AS Level and about 360 guided learning hours for a Cambridge International A Level. The number of hours a learner needs to achieve the qualification may vary according to local practice and their previous experience of the subject.

#### Availability and timetables

All Cambridge schools are allocated to one of six administrative zones. Each zone has a specific timetable.

You can view the timetable for your administrative zone at [www.cambridgeinternational.org/timetables](http://www.cambridgeinternational.org/timetables)

You can enter candidates in the June and November exam series.

Check you are using the syllabus for the year the candidate is taking the exam.

Private candidates can enter for this syllabus.

#### Combining with other syllabuses

Candidates can take this syllabus alongside other Cambridge International syllabuses in a single exam series. The only exceptions are:

- syllabuses with the same title at the same level.

Note that candidates can take AS & A Level Mathematics (9709) in the same exam series with AS & A Level Further Mathematics (9231).

#### Group awards: Cambridge AICE

Cambridge AICE (Advanced International Certificate of Education) is a group award for Cambridge International AS & A Level. It allows schools to offer a broad and balanced curriculum by recognising the achievements of learners who pass examinations in a range of different subjects.

Learn more about Cambridge AICE at [www.cambridgeinternational.org/aice](http://www.cambridgeinternational.org/aice)

## Making entries

Exams officers are responsible for submitting entries to Cambridge International. We encourage them to work closely with you to make sure they enter the right number of candidates for the right combination of syllabus components. Entry option codes and instructions for submitting entries are in the *Cambridge Guide to Making Entries*. Your exams officer has a copy of this guide.

### Exam administration

To keep our exams secure, we produce question papers for different areas of the world, known as administrative zones. We allocate all Cambridge schools to one administrative zone determined by their location. Each zone has a specific timetable. Some of our syllabuses offer candidates different assessment options. An entry option code is used to identify the components the candidate will take relevant to the administrative zone and the available assessment options.

### Support for exams officers

We know how important exams officers are to the successful running of exams. We provide them with the support they need to make your entries on time. Your exams officer will find this support, and guidance for all other phases of the Cambridge Exams Cycle, at [www.cambridgeinternational.org/eoguide](http://www.cambridgeinternational.org/eoguide)

### Retakes and carry forward

Candidates can retake Cambridge International AS Level and Cambridge International A Level as many times as they want to. Information on retake entries is at [www.cambridgeinternational.org/entries](http://www.cambridgeinternational.org/entries). To confirm what entry options are available for this syllabus, refer to the *Cambridge Guide to Making Entries* for the relevant series.

Candidates can carry forward the result of their Cambridge International AS Level assessment from one series to complete the Cambridge International A Level in a following series, subject to the rules and time limits described in the *Cambridge Handbook*.

Regulations for carrying forward entries for staged assessment (Cambridge International AS & A Level) can be found in the *Cambridge Handbook* for the relevant year of assessment at [www.cambridgeinternational.org/eoguide](http://www.cambridgeinternational.org/eoguide)

## Equality and inclusion

We have taken great care to avoid bias of any kind in the preparation of this syllabus and related assessment materials. In our effort to comply with the UK Equality Act (2010) we have taken all reasonable steps to avoid any direct and indirect discrimination.

The standard assessment arrangements may present barriers for candidates with impairments. Where a candidate is eligible, we may be able to make arrangements to enable that candidate to access assessments and receive recognition of their attainment. We do not agree access arrangements if they give candidates an unfair advantage over others or if they compromise the standards being assessed.

Information on access arrangements is in the *Cambridge Handbook* at [www.cambridgeinternational.org/eoguide](http://www.cambridgeinternational.org/eoguide)

## Language

This syllabus and the related assessment materials are available in English only.

## After the exam

### Grading and reporting

Grades A\*, A, B, C, D or E indicate the standard a candidate achieved at Cambridge International A Level. A\* is the highest and E is the lowest grade.

Grades a, b, c, d or e indicate the standard a candidate achieved at Cambridge International AS Level. 'a' is the highest and 'e' is the lowest grade.

'Ungraded' means that the candidate's performance did not meet the standard required for the lowest grade (E or e). 'Ungraded' is reported on the statement of results but not on the certificate. In specific circumstances your candidates may see one of the following letters on their statement of results:

- Q (PENDING)
- X (NO RESULT).

These letters do not appear on the certificate.

If a candidate takes a Cambridge International A Level and fails to achieve grade E or higher, a Cambridge International AS Level grade will be awarded if both of the following apply:

- the components taken for the Cambridge International A Level by the candidate in that series included all the components making up a Cambridge International AS Level
- the candidate's performance on the AS Level components was sufficient to merit the award of a Cambridge International AS Level grade.

On the statement of results and certificates, Cambridge International AS & A Levels are shown as General Certificates of Education, GCE Advanced Subsidiary Level (GCE AS Level) and GCE Advanced Level (GCE A Level).

**'Cambridge International A Levels are the 'gold standard' qualification. They are based on rigorous, academic syllabuses that are accessible to students from a wide range of abilities yet have the capacity to stretch our most able.'**

Mark Vella, Director of Studies, Auckland Grammar School, New Zealand

## How students, teachers and higher education can use the grades

### Cambridge International A Level

Assessment at Cambridge International A Level has two purposes:

- to measure learning and achievement

The assessment:

- confirms achievement and performance in relation to the knowledge, understanding and skills specified in the syllabus, to the levels described in the grade descriptions.

- to show likely future success

The outcomes:

- help predict which students are well prepared for a particular course or career and/or which students are more likely to be successful
- help students choose the most suitable course or career.

### Cambridge International AS Level

Assessment at Cambridge International AS Level has two purposes:

- to measure learning and achievement

The assessment:

- confirms achievement and performance in relation to the knowledge, understanding and skills specified in the syllabus.

- to show likely future success

The outcomes:

- help predict which students are well prepared for a particular course or career and/or which students are more likely to be successful
- help students choose the most suitable course or career
- help decide whether students part way through a Cambridge International A Level course are making enough progress to continue
- guide teaching and learning in the next stages of the Cambridge International A Level course.

## Grade descriptions

Grade descriptions are provided to give an indication of the standards of achievement candidates awarded particular grades are likely to show. Weakness in one aspect of the examination may be balanced by a better performance in some other aspect.

Grade descriptions for Cambridge International AS & A Level Further Mathematics will be published after the first assessment of the A Level in 2020. Find more information at [www.cambridgeinternational.org/alevel](http://www.cambridgeinternational.org/alevel)



## Changes to this syllabus for 2023, 2024 and 2025

The latest syllabus is version 1, published September 2020.

**You must read the whole syllabus before planning your teaching programme.**

Any textbooks endorsed to support the syllabus for examination from 2020 are still suitable for use with this syllabus.



'While studying Cambridge IGCSE and Cambridge International A Levels, students broaden their horizons through a global perspective and develop a lasting passion for learning.'

**Zhai Xiaoning**, Deputy Principal, The High School Affiliated to Renmin University of China

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